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Almost strict total positivity and a class of Hurwitz polynomials

Dimitar K. Dimitrov[a,*,1], Juan Manuel Peña[b,2]

[a] *Departamento de Ciências de Computação e Estatística, IBILCE, Universidade Estadual Paulista, Brazil*
[b] *Departamento de Matemática Aplicada, Universidad de Zaragoza, Spain*

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Abstract

We establish sufficient conditions for a matrix to be almost totally positive, thus extending a result of Craven and Csordas who proved that the corresponding conditions guarantee that a matrix is strictly totally positive. Then we apply our main result in order to obtain a new criteria for a real algebraic polynomial to be a Hurwitz one. The properties of the corresponding "extremal" Hurwitz polynomials are discussed.

Keywords: Totally positive matrix; Strictly totally positive matrix; Shadows' lemma; Hurwitz polynomial; Entire function in the Laguerre–Pólya class

1. Introduction

A real matrix is called *totally positive* (TP) if all its minors are nonnegative and *strictly totally positive* (STP) if they are positive. Many properties and a variety of applications of these matrices can be found in the book of Karlin [17] and in the comprehensive survey

* Corresponding author.
E-mail address: dimitrov@dcce.ibilce.unesp.br (D.K. Dimitrov).
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paper of Ando [1] (see also [25]). An interesting sufficient condition for strict total positivity was established by Craven and Csordas in [10]:

Theorem A (*Craven and Csordas [10], Theorem 2.2*). *Let $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ be a matrix with positive entries and*

$$a_{ij} a_{i+1,j+1} \geqslant \delta a_{i,j+1} a_{i+1,j}, \quad 1 \leqslant i, j \leqslant n-1, \tag{1}$$

where $\delta \approx 4.0795956235$ is the unique real root of $x^3 - 5x^2 + 4x - 1 = 0$. Then A is strictly totally positive.

Let us observe that (1) is far from being a necessary condition for strict total positivity. However, it is a rather simple and convenient sufficient condition because it allows the total positivity to be affirmed only by verifying (1) and the positivity of the elements of the matrix, and the inequalities (1) themselves are a condition for the 2×2 minors of A composed by consecutive rows and columns. We prove an extension of this result without the requirement that the entries of A are positive. Applications to the theory of entire function and to the Hurwitz stable polynomials are discussed. We formulate the conjecture that the smallest possible value of the constant δ to set in (1) is 4 if one considers matrices of any order and it is $4\cos^2(\pi/(n+1))$ for $n \times n$ matrices. Arguments in support of the conjecture are provided.

A special subclass of totally positive matrices, called *almost strictly totally positive* (ASTP), which include those that are strictly totally positive was introduced by Gasca et al. [13]. In order to provide the formal definition of ASTP matrices we need to introduce some notions. For $k, n \in \mathbb{N}$, $1 \leqslant k \leqslant n$, by $Q_{k,n}$ we denote the set of all increasing sequences of k natural numbers, not exceeding n. By $Q^0_{k,n}$ we shall mean the set of sequences of k *consecutive* natural numbers less than or equal to n. For a real $n \times n$ matrix A and a pair of multiindeces $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta = (\beta_1, \ldots, \beta_k)$, $\alpha, \beta \in Q_{k,n}$, we denote by $A[\alpha|\beta]$ the $k \times k$ submatrix of A composed by rows $\alpha_1, \ldots, \alpha_k$ and columns $\beta_1, \ldots, \beta_k$ of A. In particular, when $\alpha = \beta$, we set $A[\alpha] := A[\alpha|\alpha]$. Thus, a *nonsingular* matrix A of order n is called ASTP if it is totally positive and satisfies the following property: a minor of A formed by consecutive rows and consecutive columns is positive if and only if all its diagonal entries are positive. Equivalently,

$$\det\, A[\alpha|\beta] > 0 \Longleftrightarrow a_{\alpha_\nu, \beta_\nu} > 0, \quad \nu = 1, \ldots, k \tag{2}$$

and it must hold for any $\alpha, \beta \in Q^0_{k,n}$. It was proved in [13] that, if A is ASTP, then (2) holds not only for the multiindeces in $Q^0_{k,n}$ but for any $\alpha, \beta \in Q_{k,n}$. Consequently, for this type of matrices we know exactly the minors which are positive and the ones which are zero. Characterization of ASTP matrices by means of the Neville elimination, in terms of their *LU*-factorizations, as a product of bidiagonal elementary matrices, as well as in terms of positivity of certain minors determined through the so-called zero patterns, were provided in [14].

Important ASTP matrices are the Hurwitz matrices [3,19] and the B-splines collocation matrices [4]. Some examples of applications of these matrices in Approximation Theory can be seen in [6]. Recently Garloff [12] proved that, when A_1 and A_2 are ASTP matrices with $A_1 \prec A_2$, where $\prec$ denotes the so-called chequerboard partial ordering, so are all A satisfying $A_1 \prec A \prec A_2$.

It is known that no nonsingular TP matrix can have zeros as diagonal entries [1, Corollary 3.8]. Then we can deduce from the shadows' lemma (see [5, Lemma A]) that, if $A = (a_{ij})$ is a nonsingular $n \times n$ TP matrix, then

$$\begin{array}{ll} a_{ii} > 0 & \text{for } i = 1, \dots, n \\ \text{if } a_{ij} = 0,\ i > j & \text{then } a_{hk} = 0 \text{ for all } h \geqslant i \text{ and } k \leqslant j \\ \text{If } a_{ij} = 0,\ i < j & \text{then } a_{hk} = 0 \text{ for all } h \leqslant i \text{ and } k \geqslant j. \end{array} \tag{3}$$

Before we state our extension of Theorem A to the class of ASTP matrices, recall that a matrix is called nonnegative (positive) if all its entries are nonnegative (positive).

Theorem 1. *Let $A = (a_{ij})$ be a nonnegative $n \times n$ matrix satisfying* (3). *Assume that, for any $1 \leqslant i, j \leqslant n-1$, the following condition holds*:

$$\text{if } a_{ij}a_{i+1,j+1} > 0, \quad \text{then } a_{ij}a_{i+1,j+1} \geqslant \delta a_{i,j+1}a_{i+1,j}, \tag{4}$$

where δ is given in Theorem A. Then A is TP. Moreover, if the second inequality in (4) *is strict, then A is nonsingular ASTP.*

One of the consequences of this result is for the theory of entire functions with real zeros. A real entire function $\psi(x)$ is said to belong to the *Laguerre–Pólya class*, written $\psi \in \mathcal{L}-\mathcal{P}$, if $\psi(x)$ can be represented in the form

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\omega} (1 + x/x_k) e^{-x/x_k}, \quad (0 \leqslant \omega \leqslant \infty), \tag{5}$$

where c, β, x_k are real, $\alpha \geqslant 0$, m is a nonnegative integer, $\sum x_k^{-2} < \infty$ and where the canonical product reduces to 1 when $\omega = 0$. Pólya and Schur [28] called the real entire function $\varphi(x)$ a function of *type I* in the Laguerre–Pólya class, written $\varphi \in \mathcal{L}-\mathcal{P}I$, if $\varphi(x)$ or $\varphi(-x)$ can be represented in the form

$$\varphi(x) = cx^m e^{\sigma x} \prod_{k=1}^{\omega} (1 + x/x_k), \quad (0 \leqslant \omega \leqslant \infty), \tag{6}$$

where c is real, $\sigma \geqslant 0$, m is a nonnegative integer, $x_k > 0$, and $\sum 1/x_k < \infty$. It is clear that $\mathcal{L}-\mathcal{P}I \subset \mathcal{L}-\mathcal{P}$. The importance of the Laguerre–Pólya class $\mathcal{L}-\mathcal{P}$ ($\mathcal{L}-\mathcal{P}I$, respectively) is revealed by the fact that the functions in $\mathcal{L}-\mathcal{P}$ ($\mathcal{L}-\mathcal{P}I$), and only these, are the uniform limits, on compact subsets of $\mathbb{C}$, of polynomials with only real (nonpositive) zeros [21, Chapter VIII]. Pólya and Schur [28] observed that, if a function

$$\varphi(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \tag{7}$$

is in $\mathcal{L}-\mathcal{P}$ and its Maclaurin coefficients γ_k are nonnegative, then $\varphi \in \mathcal{L}-\mathcal{P}I$. In the same fundamental paper [28] Pólya and Schur introduced the notion *multiplier sequence* calling by this any sequence $\{\gamma_k\}_0^\infty$ of Maclaurin coefficients of a function in $\mathcal{L}-\mathcal{P}I$. The reader may consult [8,9], [21, Chapter VIII], [24, Kapitel II], [27] and the references therein for more information about the properties of the functions in the Laguerre–Pólya class. We only mention that a necessary condition for an entire function $\varphi(x)$, defined by (7), to belong to $\mathcal{L}-\mathcal{P}I$ is that the following Turán inequalities

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geqslant 0, \quad k = 1, 2, \ldots,$$

hold. As an immediate consequence of Theorem 1, we obtain the following sufficient conditions of a function to be in $\mathcal{L}-\mathcal{P}I$.

Corollary 2. *If the coefficients γ_k in the formal power series $\sum_{k=0}^\infty \gamma_k x^k/k!$ are positive and satisfy*

$$\gamma_k^2 - \frac{k}{k+1}\delta\gamma_{k-1}\gamma_{k+1} \geqslant 0, \quad k = 1, 2, \ldots, \tag{8}$$

then it represents an entire function $\varphi(x)$ of genus 0 and $\varphi \in \mathcal{L}-\mathcal{P}I$. In particular, if the coefficients γ_k of the polynomial $p(z) = \sum_{k=0}^n \gamma_k x^k/k!$ are positive and satisfy (8) for $k = 1, \ldots, n-1$, then all the zeros of $p(z)$ are real and negative.

While we were not able to prove Theorem 1 with the best possible value 4 instead of the constant δ and we provide a short proof of Corollary 2 only for the sake of completeness and as an illustrative application of Theorem 1, results corresponding to Corollary 2, already with the constant 4 instead of δ, are known. In 1923 Hutchinson [16], extending the work of Petrovitch [26] and Hardy [15], proved the following beautiful result for entire function

$$f(x) = \sum_{k=0}^\infty a_k x^k,$$

whose coefficients a_k are given by $a_0 = 1$ and

$$a_k = \frac{1}{b_1 b_2 \cdots b_k}, \quad k = 1, 2, \ldots.$$

Theorem B ([16, Theorem A. p. 327]). *The relations*

$$b_k \geqslant 4 b_{k-1}, \quad k = 2, 3, \ldots, \tag{9}$$

are the necessary and sufficient conditions that the series $f(x)$ may have the properties:
1. *The zeros of $f(x)$ are real, simple and negative; and*
2. *The zeros of any polynomial $a_m x^m + \cdots + a_n x^n$ formed by taking any number of consecutive terms of $f(x)$ are all real, simple, and negative (excepting $x = 0$).*

It is worth mentioning a small gap in Hutchinson's proof. Theorem B is correct either without the statement for simplicity of the zeros of the polynomials in part 2 or if we substitute (9) by the corresponding strict inequalities. Indeed, if we take $f(x) = 1 + x + x^2/4 + \cdots$, then the partial sum $f_2(x) = 1 + x + x^2/4$ has a double root at -2. Observe that the inequalities (9) are equivalent to the inequalities $a_k^2 - 4a_{k-1}a_{k+1} \geqslant 0$ for the Maclaurin coefficients of $f(x) = \sum_{k=0}^{\infty} a_k x^k$, or to $\gamma_k^2 - 4\frac{k}{k+1}\gamma_{k-1}\gamma_{k+1} \geqslant 0$ if $f(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k!$. Craven and Csordas [9] proved extensions of Hutchinson's result.

Recently Kurtz [20] considered only the polynomial case, and proved that, if $n \geqslant 2$ and the coefficients a_k of the polynomial

$$P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$$

are all positive and satisfy the inequalities

$$a_k^2 - 4a_{k-1}a_{k+1} > 0, \quad k = 1, \ldots, n-1, \tag{10}$$

then all the zeros of $P_n(x)$ are negative and distinct. Moreover, Kurtz observed the sharpness of (10) showing that, for any given $\varepsilon > 0$ and $n \in \mathbb{N},\ n \geqslant 2$, there exists a polynomial of degree n, which has some nonreal zeros and whose coefficients are positive and satisfy $a_k^2 - (4-\varepsilon)a_{k-1}a_{k+1} > 0$ for $k = 1, \ldots, n-1$.

However, if one considers entire functions with positive coefficients, i.e. when property 2 in Hutchinson's theorem is omitted, then the constant α in the inequalities

$$a_k^2 - \alpha a_{k-1}a_{k+1} > 0, \quad k = 1, 2, \ldots$$

for its Maclaurin coefficients may have somehow smaller value than 4. In a very recent paper Katkova et al. [18], studied in details the extremal value of the constant α as well the properties of the corresponding extremal entire function, the one for which inequalities reduce to equalities.

Another application of Theorem 1 concerns the so-called Hurwitz (stable) polynomials, namely, polynomials $f(z) = c_n z^n + c_{n-1}z^{n-1} + \cdots + c_0$ with real coefficients c_j, whose zeros have negative real parts. We refer to [11, Chapter 15], [23, Chapter 9] for comprehensive information on the stability theory. We only mention that a necessary condition for a polynomial $f(z)$ with positive leading coefficient to be Hurwitz one is that all its coefficients are positive.

Theorem 3. *Let δ be defined as in Theorem A. If the coefficients of*

$$f(z) = c_n z^n + c_{n-1}z^{n-1} + \cdots + c_0$$

are positive and satisfy the inequalities

$$c_k c_{k+1} \geqslant \delta\, c_{k-1}c_{k+2} \quad \text{for} \quad k = 1, \ldots, n-2, \tag{11}$$

then $f(z)$ is a Hurwitz polynomial. In particular, the conclusion is true if

$$c_k^2 \geqslant \sqrt{\delta}\, c_{k-1}c_{k+1} \quad \text{for} \quad k = 1, \ldots, n-1. \tag{12}$$

Observe that inequalities (12) imply that the zeros of $f(z)$ have zeros with negative real parts while the similar but stronger requirements (10) guarantee that these zeros are real, negative and distinct. We refer to [29,30] for some necessary conditions for a real polynomial to be stable.

2. Proof of the main result

Proof of Theorem 1. Given the matrix A satisfying (4), let us construct an $n \times n$ positive matrix $B = (b_{ij})$ such that, for $1 \leqslant i, j \leqslant n-1$,

$$b_{ij}b_{i+1,j+1} \geqslant \delta b_{i,j+1}b_{i+1,j}. \tag{13}$$

For any (i, j) such that $a_{ij} \neq 0$, we define $b_{ij} := a_{ij}$.

If $\{(i, j) | a_{ij} = 0, |i-j| = 1\} = \{(i_1^1, j_1^1), \ldots, (i_{r_1}^1, j_{r_1}^1)\}$ with $i_1^1 \leqslant i_2^1 \leqslant \cdots \leqslant i_{r_1}^1$ and, if $i_k^1 = i_{k+1}^1$ for some k, then $j_k^1 < j_{k+1}^1$, clearly we can choose positive numbers $b_{i_1^1, j_1^1}, \ldots, b_{i_{r_1}^1, j_{r_1}^1}$ such that (13) holds for all $1 \leqslant i = j \leqslant n-1$. Let us now continue to fill in the lower triangular part of A. If $\{(i, j) | a_{ij} = 0, i-j = 2\} = \{(i_1^2, j_1^2), \ldots, (i_{r_2}^2, j_{r_2}^2)\}$ with $i_1^2 < i_2^2 < \cdots < i_{r_2}^2$, then we can choose positive numbers $b_{i_1^2, j_1^2}, \ldots, b_{i_{r_2}^2, j_{r_2}^2}$ such that (13) holds for all $1 \leqslant i, j \leqslant n-1$ with $i-j=1$. Analogously, we can iterate the previous procedure until we obtain all elements $b_{ij} > 0$ (with $i \geqslant j$) satisfying (13) for $1 \leqslant i, j \leqslant n-1$ and $i \geqslant j$. In a similar way, we can fill in the upper triangular part of A in order to obtain a positive matrix B satisfying (13) for $1 \leqslant i, j \leqslant n-1$.

Let $0 < \varepsilon < 1$ and let B_ε be the matrix obtained from B by replacing the elements $b_{i_k^s, j_k^s}$ by the elements $b_{i_k^s, j_k^s}\varepsilon^{2^{s-1}}$. Then it can be checked that the entries of B satisfy a condition analogous to (13). Since B_ε is positive and satisfies (13), we deduce from Theorem A that B_ε is an STP matrix for each ε. Taking limits as $\varepsilon \to 0$, we deduce that the matrices B_ε converge to A. Since the set of TP matrices is closed, we conclude that A is TP.

Now, suppose that the second inequality in (4) is strict and let us prove that A is nonsingular ASTP. For this purpose, it is sufficient to get a contradiction after assuming that there exists an $h \times h$ submatrix $C = (c_{ij})$ formed by consecutive rows and columns of A and whose positive diagonal entries are positive and $\det C = 0$. Let $h > 1$ be the least integer satisfying the previous property. Since A is nonnegative and satisfies (4) with the second inequality strict, we can find $\tau > 0$ such that

$$(c_{11} - \tau)c_{22} > \delta c_{12}c_{21}.$$

Let C_τ be the matrix with the entries of C but with $c_{11} - \tau$ instead of c_{11}. Since C is a submatrix of A formed by consecutive rows and columns and its diagonal entries are positive, we deduce that C, and so C_τ too, satisfy the hypotheses of A. Thus, by the first part of the proof, C_τ is TP, and so $\det C_\tau \geqslant 0$. Taking into account that $\det C_\tau[2, \ldots, h] = \det C[2, \ldots, h] > 0$ by our choice of h, we can deduce by the expansion of $\det C_\tau$ on its first row that $\det C_\tau < \det C = 0$: a contradiction which proves the result. □

3. The smallest value of the constant δ

Before we prove the applications of Theorem 1 to entire functions and to stable polynomials, we shall discuss in this section the smallest possible value of the constant δ in Theorems A and 1. First, we consider the case when the dimension of the matrix is fixed.

Theorem 4. *Let $n \in \mathbb{N}$, $n \geqslant 2$. Then, for any $\varepsilon > 0$ there exist an $n \times n$ positive matrix $A_{n,\varepsilon} = (a_{ij})$ for which*

$$a_{ij}a_{i+1,j+1} \geqslant 4(1-\varepsilon)\cos^2(\pi/(n+1))\, a_{i+1,j}a_{i,j+1}, \quad 1 \leqslant i, j \leqslant n-1, \tag{14}$$

but $A_{n,\varepsilon}$ is not STP.

Proof. Consider the $n \times n$ Jacobi matrix

$$A_n(\varepsilon,\kappa) = \begin{pmatrix} \sqrt{1-\varepsilon}\,\kappa & 1/2 & & & O \\ 1/2 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1/2 \\ O & & & 1/2 & \sqrt{1-\varepsilon}\,\kappa \end{pmatrix},$$

where ε is any real number with $0 < \varepsilon < 1$ and let $Q_m(x)$ be the characteristic polynomial of $A_m(\varepsilon,\kappa)$, $m \geqslant 1$. Then the sequence of polynomials $\{Q_m(x)\}_{m=0}^{\infty}$ is generated by the three term recurrence relation

$$\begin{aligned} Q_0(x) &:= 1; \\ Q_1(x) &= \sqrt{1-\varepsilon}\,\kappa - x; \\ Q_{m+1}(x) &= (\sqrt{1-\varepsilon}\,\kappa - x)Q_m(x) - (1/4)Q_{m-1}(x), \quad m = 1, 2, \ldots. \end{aligned}$$

On the other hand, the Chebyshev polynomials of the second kind $U_m(x)$, defined by $U_m(\cos\theta) = \sin((m+1)\theta)/\sin\theta$, satisfy the recurrence relation $U_{m+1}(x) = 2xU_m(x) - U_{m-1}(x)$, $m = 1, 2, \ldots$, with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. Thus, the characteristic polynomial of $A_n(\varepsilon,\kappa)$ is the Chebyshev polynomial $U_n(x)$ with shifted argument,

$$Q_n(x) = (-1/2)^n U_n(x - \sqrt{1-\varepsilon}\,\kappa).$$

Then, since the zeros of $U_n(x)$ are $\cos(k\pi/(n+1))$, $k = 1, \ldots, n$, those of $Q_n(x)$ are $\zeta_k = \sqrt{1-\varepsilon}\,\kappa + \cos(k\pi/(n+1))$. Therefore, for $\kappa = \kappa_n := \cos(\pi/(n+1))$, if $\varepsilon > 0$, at least the smallest zero ζ_n of $Q_n(x)$ is negative. Hence, for $\kappa = \kappa_n$, the matrix $A_n(\varepsilon,\kappa_n)$ is not positive definite, and then it is not a TP matrix. On the other hand, the inequalities (14) for $i = j$, reduce to equalities for this matrix.

Let μ be any positive number with

$$\mu < (1-\varepsilon)^{-1/2}\kappa_n^{-1}. \tag{15}$$

Set $k := |i-j|$ and let us define the $n \times n$ matrix $A_n(\varepsilon,\kappa_n,\mu)$ whose elements a_{ij} coincide with those of $A_n(\varepsilon,\kappa_n)$ when $k \leqslant 1$ and are given by $a_{ij} := \mu^{k-1}/2^{k^2}$ when $k \geqslant 2$. The matrix $A_n(\varepsilon,\kappa_n,\mu)$ is positive. As it was pointed out, (14) holds for $k = 0$. The above requirements on μ guarantee that it holds for $k = 1$. For $k \geqslant 2$ (14) is obviously satisfied even for any real μ.

Observe that $\lim_{\mu\to 0} A_n(\varepsilon,\kappa_n,\mu) = A_n(\varepsilon,\kappa_n)$. Since the set of TP matrices is closed, if the matrices $A_n(\varepsilon,\kappa_n,\mu)$ were STP for all values of μ which satisfy (15), then $A_n(\varepsilon,\kappa_n)$

would be a TP matrix. This contradiction implies that there exist positive matrices $A_n(\varepsilon, \kappa_n, \mu)$ satisfying (14) which are not STP matrices and the result follows. □

Letting n to tend to infinity, we see that the bound δ of Theorem A cannot be reduced to less than 4 when we consider matrices of any order n.

Corollary 5. *For any $\varepsilon > 0$ there exist $n \in \mathbb{N}$, $n \geqslant 2$, and an $n \times n$ positive matrix $A = (a_{ij})$ such that*

$$a_{ij}a_{i+1,j+1} \geqslant 4(1-\varepsilon)a_{i+1,j}a_{i,j+1},$$

and which is not STP.

We strongly believe that the matrices constructed in the proof of Theorem 4 are in some sense the extremal ones and we venture to suggest the following conjecture.

Conjecture 6. *Let $A = (a_{ij})$ be a nonnegative $n \times n$ matrix satisfying* (3). *Assume that, for any $1 \leqslant i, j \leqslant n-1$, the following condition holds*:

$$\text{if } a_{ij}a_{i+1,j+1} > 0, \quad \text{then } a_{ij}a_{i+1,j+1} > 4\cos^2(\pi/(n+1))a_{i,j+1}a_{i+1,j}. \tag{16}$$

Then A is nonsingular ASTP.

In particular, if $A = (a_{ij})$ is a positive $n \times n$ matrix whose entries satisfy

$$a_{ij}a_{i+1,j+1} > 4\cos^2(\pi/(n+1))a_{i,j+1}a_{i+1,j}, \quad 1 \leqslant i, j \leqslant n-1,$$

then A is strictly totally positive.

Needless to say, when we consider matrices of any order, the above conditions reduce to

$$a_{ij}a_{i+1,j+1} \geqslant 4a_{i,j+1}a_{i+1,j}$$

and, as seen from Corollary 5, the constant 4 cannot be reduced.

4. Entire functions in the Laguerre–Pólya class and Hurwitz polynomials

We begin this section with some additional information about entire functions in the Laguerre–Pólya class. Recall that an infinite sequence $\{a_k\}_{k=0}^{\infty}$ is said to be *totally positive* (or *Pólya frequency sequence*) if $\sum_{k=0}^{\infty} a_k x^k$ is an entire function and the infinite upper triangular matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ddots & \ddots & \ddots \\ & a_0 & a_1 & a_2 & \ddots & \ddots \\ & & a_0 & a_1 & \ddots & \ddots \\ & & & a_0 & \ddots & \ddots \\ & & & & \ddots & \ddots \end{pmatrix} \tag{17}$$

is totally positive. Corollary 2 is an immediate consequence of Theorem 1 and the following characterization of functions in the Laguerre–Pólya class with nonnegative Maclaurin coefficients in terms of totally positive sequences, due to Aisen et al. [2]:

Theorem C. *The real entire function $\varphi(x) = \sum_{k=0}^{\infty} a_k x^k$ with nonnegative coefficients a_k is in the Laguerre–Pólya class if and only if the sequence $\{a_k\}_{k=0}^{\infty}$ is totally positive.*

Indeed, the Maclaurin coefficients of

$$\varphi(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}, \tag{18}$$

satisfy inequalities

$$a_k^2 \geqslant \delta a_{k-1} a_{k+1}, \quad k = 1, 2, \ldots, \tag{19}$$

which are equivalent to (8) and so, by Theorem 1, the sequence $\{a_k\}_{k=0}^{\infty}$ is totally positive provided $\varphi(x)$ is an entire function. Thus, in order to prove Corollary 2 we only need to prove that $\varphi(x)$ is an entire function of order zero. We shall prove that, if a positive sequence $\{a_k\}_{k=0}^{\infty}$ satisfies inequalities (19), then

$$a_k \leqslant \frac{a_1^k}{a_0^{k-1}}\, \delta^{-k(k-1)/2} \quad \text{for} \quad k \geqslant 2. \tag{20}$$

If we set $b_k = a_{k+1}/a_k$, then the inequalities (19) are equivalent to the inequalities $b_k \leqslant \delta^{-1} b_{k-1}$. These immediately yield

$$b_k \leqslant (a_1/a_0)\delta^{-k}. \tag{21}$$

Now, we are in a position to prove (20) by induction with respect to k. Inequality (20) for $k = 2$ is exactly (19) for $k = 1$. Suppose that (20) holds for some natural number k. Then, the induction passage follows from the following simple chain of inequalities where we use (19), (21) and the induction hypothesis (20):

$$a_{k+1} \leqslant \delta^{-1} \frac{a_k^2}{a_{k-1}} = \delta^{-1} b_{k-1} a_k \leqslant \delta^{-1} \frac{a_1}{a_0} \delta^{-k+1} \frac{a_1^k}{a_0^{k-1}} \delta^{-k(k-1)/2} = \frac{a_1^{k+1}}{a_0^k}\, \delta^{-k(k+1)/2}.$$

It is well known that the function $\varphi(x)$ of the form (18) is entire if its coefficients satisfy $\lim_{n\to\infty} |a_n|^{1/n} = 0$ and in this case the order ρ of $\varphi(x)$ is given by (see [22, Lecture1])

$$\rho = \lim\ \sup_{n\to\infty} \frac{n \log n}{\log(1/|a_n|)}.$$

Observe that the inequalities (20) are equivalent to

$$\alpha_k := a_k/a_0 \leqslant C^k \delta^{-k(k-1)/2},$$

where $C = a_1/a_0$. Then

$$\frac{n \log n}{\log(1/|\alpha_n|)} \leqslant \frac{\log n}{(n-1)\log \delta^{1/2} - \log C}.$$

Since the order of an entire function does not depend on multiplication by a constant, then $\varphi(x)$ is an entire function of order zero.

The extremal entire function for which the inequalities in Hutchinson's theorem reduce to equalities turns out to be an interesting one. If we fix $a_0 = 1$ and $a_1 = \frac{1}{2}$, then obviously we have equalities in (9) (or, equivalently, $a_k^2 = 4a_{k-1}a_{k+1}$) provided $a_n = 2^{-n^2}$. Then the requirements of Theorem B will be satisfied if $a_n = q^{n^2}$, $n = 0, 1, \ldots$, and $q \leqslant \frac{1}{2}$. Thus we conclude that

$$\sum_{n=0}^{\infty} q^{n^2} x^n, \tag{22}$$

is an entire function of order zero which belongs to $\mathcal{L}-\mathcal{P}I$ whenever $0 < q \leqslant \frac{1}{2}$. Katkova et al. [18, Theorem 4] proved the existence of a constant $q_\infty \approx 0.556415$, such that the function (22) has only real zeros if and only if $q \leqslant q_\infty$. It is worth mentioning that it was proved recently in [7] that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{n!} x^n,$$

is in $\mathcal{L}-\mathcal{P}$ if $|q| < 1$. In fact, the equivalent fact that the sequence $\{q^{n^2}\}$ is a multiplier (or zero-increasing) sequence for $|q| < 1$ was pointed out in [7], while the result in [18] shows $\{n! q^{n^2}\}$ is a multiplier sequence if and only if $0 < q \leqslant q_\infty$.

The proof of Theorem 3 is an immediate consequence of Theorem 1 and a result of Hurwitz. Here we only provide the necessary definitions and formulate the Hurwitz theorem. With the polynomial

$$f(z) = c_n z^n + c_{n-1} z^{n-1} + c_{n-2} z^{n-2} + c_{n-3} z^{n-3} + \cdots + c_0,$$

we associate the Hurwitz matrix which is formed as follows. Set $c_{-1} = c_{-2} = \cdots = 0$ and construct the two line block

$$\begin{pmatrix} c_{n-1} & c_{n-3} & \cdots \\ c_n & c_{n-2} & \cdots \end{pmatrix},$$

where the first line contains c_{n-2k-1}, $k = 0, 1, \ldots,$ and the second line is composed by the coefficients c_{n-2k}, $k = 0, 1, \ldots,$ of $f(z)$. Then, the Hurwitz matrix $H(f)$ of $f(z)$ is composed by the above block in its first two lines, the next two lines of $H(f)$ contain the same block shifted one position to the right, the fifth and the sixth lines contain this block shifted two positions to the right, and so forth. Thus

$$H(f) = \begin{pmatrix} c_{n-1} & c_{n-3} & c_{n-5} & \cdots & 0 \\ c_n & c_{n-2} & c_{n-4} & \cdots & 0 \\ 0 & c_{n-1} & c_{n-3} & \cdots & 0 \\ 0 & c_n & c_{n-2} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \end{pmatrix}.$$

The following is the Hurwitz theorem which is sometimes called the Routh–Hurwitz criterion.

Theorem D. *The polynomial $f(z)$ with $c_n > 0$ is stable if and only if the first n principal minors of the corresponding Hurwitz matrix $H(f)$ are positive.*

Since the matrix $H(f)$ satisfies the requirements of the shadows' lemma, then the fact that $f(z)$ is a Hurwitz polynomial in Theorem 3 does follow immediately from Theorem 1. To complete the proof of Theorem 3, it remains to observe that the conditions (12) imply (11).

Interesting examples of Hurwitz polynomials are those for which the inequalities (12) reduce to equalities. Let δ be defined as in Theorem A and $\tilde{q} = \delta^{-1/2} \approx 0.495098$. It follows from Theorem 3 that the polynomials

$$f_n(z) = \sum_{k=0}^{n} q^{k^2} x^k$$

are stable when $q \leqslant \tilde{q}^{1/2} \approx 0.703632$ and, when $q = \tilde{q}^{1/2}$, (12) reduce to equalities for the coefficients of $f_n(z)$. On the other hand, motivated by the results in Section 3, we believe that $f_n(z)$ are still stable for $q \leqslant 1/\sqrt{2} \approx 0.70710678$ and even for larger values of q. On the other hand, Theorem 4 in [18] implies that the same polynomials have only real and negative zeros when $q \leqslant q_\infty \approx 0.556415$, at least for large values of $n \in \mathbb{N}$. These consequences of our results suggest a challenging question about the behaviour of the zeros of $f_n(z)$. Given a positive integer n, which are the largest values of the constants m_n and M_n, such that the zeros of $f_n(z)$ are:

- real and negative when $q \in (0, m_n]$?
- with negative real parts when $q \in (0, M_n]$?

Obviously $m_n < M_n$, Theorem 4 in [18] and Theorem 3 in the present paper show that these constants satisfy the inequalities $q_\infty < m_n$ and $\tilde{q}^{1/2} < M_n$, and obviously $M_n < 1$ for $n \geqslant 4$. The polynomial $f_2(z)$ is stable for any positive q and it has real zeros if and only if $q \leqslant \frac{1}{2}$ which means that $m_2 = \frac{1}{2}$. For $n = 3$ we have $m_3 = 1/\sqrt{3}$ and $M_3 = 1$. Do m_n and M_n maintain a monotonic behavior and do they converge as n goes to infinity? In particular, is it true that $m_n \to q_\infty$ as n goes to infinity?

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